Note on an n-dimensional Pythagorean theorem

Sergio A. Alvarez
Center for Nonlinear Analysis and
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890

Abstract

A famous theorem in Euclidean geometry often attributed to the Greek thinker Pythagoras of Samos (6th century, B.C.) states that if one of the angles of a planar triangle is a right angle, then the square of the length of the side opposite the right angle equals the sum of the squares of the lengths of the sides which form the right angle. There are less commonly known higher-dimensional versions of this theorem which relate the areas of the faces of a simplex having one "orthogonal vertex" by analogous sums-of-squares identities. In this note I state and prove one such result, hoping that students of mathematics will become better acquainted with it.

Introduction

> "Geometry has two great treasures. One is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."
>
> *J. Kepler ([2], p. 58.)*

In high school geometry one learns the *Pythagorean theorem*, stating that "the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides" (see [6], Propositions 47 and 48). This fact was known to the Babylonians over one thousand years prior to the time of Pythagoras, as numerous clay tablets such as the famous Plimpton 322 tablet show [2], [8], [13], [16]. In fact, [4] contains a reference to a tablet dating back to 2600 B.C. on which an illustrated example of the Pythagorean theorem is given (curiously, there is no evidence suggesting that the ancient Egyptians were aware of the theorem, [7]). Hundreds of proofs of this beautiful fact are known; see for example the reference [12]. There is little doubt that the theorem is one of the most fundamental of mathematical facts.

Recently, I was beholding the wonder of this result, the fact that the mere requirement of orthogonality of one of the angles implies that such a simple quadratic relation must be satisfied by the lengths of the sides, and it occurred to me that it would be no more surprising if an analogous result held in higher dimensions. This turned out to be true, and is the subject of the present note. After showing the generalized result to several colleagues and asking them if they could provide a reference for a fact which should obviously be widely known, it became clear to me that, incredibly, comparatively few mathematicians are aware of it. Boyer ([2], 14.10, p. 288) attributes a three-dimensional generalization of the classical Pythagorean theorem to Fibonacci in his early 13th century work *Practica Geometriae*. Guggenheimer [9] suggests that the n-dimensional result is due to J. P. de Gua de Malves, in *Histoire de l'Académie des Sciences pour l'année 1783*, p. 375, Paris, 1786. The n-dimensional statement that I will present below is in fact a special case of a result described by Monge and Hachette in an early 19th century treatise on analytic geometry (see [2], 22.8, p. 533). Various versions of the result have been found anew numerous times; a Web search returned, among others, the papers [1], [3], [5], [10], [11], [14], [15]. I hope that the present note will further promote increased recognition of this beautiful and simple result.

1 A three-dimensional Pythagorean theorem

Let us begin with a three-dimensional version of the Pythagorean theorem. Instead of a triangle, consider a *three-dimensional simplex*, also known as a *tetrahedron*. This is the smallest convex region of three-dimensional Euclidean space containing four given points not lying in any one plane. Another way of describing a tetrahedron is to give instructions for its construction, as follows. Start with a planar triangle. Choose a point not lying in the plane of the triangle. Then the tetrahedron consists of all points lying on the line segment joining the chosen point with any point of the triangle.

We will not consider arbitrary tetrahedrons. Rather, we will restrict our attention to the analogs of right triangles. We will call these *orthogonal tetrahedrons.* An orthogonal tetrahedron is any tetrahedron which has a vertex at which three faces meet at right angles to each other. In terms of the constructive definition of a tetrahedron given above, this means that one should be able to start with a right triangle and that the line segment joining the non-coplanar point with the right angle of the triangle will then be perpendicular to the plane of the triangle.

In order to simplify our subsequent statements, we introduce some terminology. In an orthogonal tetrahedron, we refer to the the three faces which meet orthogonally as the *orthogonal faces*, and we refer to the remaining face as the *opposing face*.

We can now state a three-dimensional version of the Pythagorean theorem. The result is a direct analog of the classical result, in which the right triangle has been replaced by an orthogonal tetrahedron, and the lengths of the sides of the triangle have been replaced by the areas of the faces of the tetrahedron.

Theorem 1.1. *Let A be the area of the opposing face, and let A_1, A_2, A_3 be the areas of the orthogonal faces of a given orthogonal tetrahedron. Then the following relation holds:*

$$A^2 = A_1^2 + A_2^2 + A_3^2 \tag{1}$$

Proof. We work in Euclidean coordinates. Since the tetrahedron of the statement is orthogonal, the three vertices of the opposing face may be represented by scalar multiples of the three mutually orthogonal basis vectors e_1, e_2, e_3:

$$V_1 = \ell_1 e_1, \quad V_2 = \ell_2 e_2, \quad V_3 = \ell_3 e_3 \tag{2}$$

The edges joining vertex V_1 with vertices V_2 and V_3 are then represented, respectively, by the difference vectors

$$E_{1,2} = \ell_2 e_2 - \ell_1 e_1, \ E_{1,3} = \ell_3 e_3 - \ell_1 e_1 \tag{3}$$

The area A of the opposing face equals half the area of the parallelogram determined by edges $E_{1,2}$ and $E_{1,3}$. By standard vector analysis, we therefore have the cross product identity:

$$A = \frac{1}{2}|(\ell_2 e_2 - \ell_1 e_1) \times (\ell_3 e_3 - \ell_1 e_1)| \tag{4}$$

Direct computation now yields:

$$A^2 = \frac{1}{4}\left((\ell_1\ell_2)^2 + (\ell_1\ell_3)^2 + (\ell_2\ell_3)^2\right) \tag{5}$$

But since the orthogonal faces of the tetrahedron are right triangles, their areas are simply one-half of the product of the lengths of their sides:

$$A_1^2 = \frac{1}{4}(\ell_1\ell_2)^2, \quad A_2^2 = \frac{1}{4}(\ell_1\ell_3)^2, \quad A_3^2 = \frac{1}{4}(\ell_2\ell_3)^2 \tag{6}$$

We therefore see that the right-hand side of Eq. 5 is precisely the sum of the squares of the areas of the orthogonal faces, as we wanted to show.

2 Extension to higher dimensions

Next, we address the n-dimensional case. The objects replacing right triangles in this context are n-dimensional simplices. A 0-dimensional simplex is a point. A 1-dimensional simplex is a nontrivial line segment. A 2-dimensional simplex is a nondegenerate triangle. A 3-dimensional simplex is a nondegenerate tetrahedron. An n-dimensional simplex is the set of points lying on any line segment joining any point of a given $(n-1)$-dimensional simplex with a given point lying outside the $(n-1)$-dimensional convex span of the $(n-1)$-dimensional simplex. The condition replacing the requirement of a right angle is the orthogonality condition that the $(n-1)$-dimensional faces of a given n-dimensional simplex meet orthogonally at one of the vertices of the simplex. As before, we refer to the face opposite the orthogonal vertex as the opposing face, and we refer to the remaining faces as the orthogonal faces.

Theorem 2.1. *Let A be the area of the opposing face, and let $A_1, A_2, \ldots A_n$ be the areas of the orthogonal faces of a given orthogonal n-simplex. Then the following relation holds:*

$$A^2 = \sum_{j=1}^{n} A_j^2 \tag{7}$$

Theorem 2.1 may be proved by a technique analogous to that employed above in the 3-dimensional case, by expressing the area of the opposing face in terms of determinant identities related to higher-dimensional analogs of the three-dimensional cross product. However, there is a simpler proof, pointed out to the author by Luc Tartar [17], which is the one we have chosen to present here.

Proof. We deduce the area A of the opposing face by computing the *volume* of the entire simplex. By the slicing method used to compute volumes in integral calculus, we know that the volume of the simplex equals $1/n$ times the volume of the parallelipiped determined by the orthogonal faces of the simplex:

$$V = \frac{1}{n}Ad \tag{8}$$

Here, d is the distance between the orthogonal vertex and the opposing face. From Eq. 8, we can find the area A if we know the volume nV and the distance d. But the volume nV is simply the product of the lengths of the edges of the simplex:

$$nV = \Pi_{j=1}^{n}\ell_j \tag{9}$$

And the distance d may be found quite simply. Namely, working in Euclidean coordinates as in the 3-dimensional case, the equation of the plane of the opposing face is:

$$\sum_{j=1}^{n} \frac{x_j}{\ell_j} = 1 \tag{10}$$

The vector with j-th coordinate equal to the reciprocal of the length ℓ_j is normal to the opposing face. Since the distance d is measured in the direction of this normal vector, we see that the endpoint of the vector equal to d times the *normalized* normal vector must lie in the plane of the opposing face, and by Eq. 10 we must therefore have:

$$\sum_{j=1}^{n} \frac{d\frac{1}{\ell_j}}{\ell_j} = \left(\sum_{j=1}^{n} \frac{1}{\ell_j^2} \right)^{1/2} \tag{11}$$

Thus, we infer that

$$d = \left(\sum_{j=1}^{n} \frac{1}{\ell_j^2} \right)^{-1/2} \tag{12}$$

From Eqs. 9 and 12 together, we find that the square A^2 of the area of the opposing face satisfies:

$$A^2 = \left(\Pi_{j=1}^{n} \ell_j \right)^2 \sum_{k=1}^{n} \frac{1}{\ell_k^2} = \sum_{k=1}^{n} \left(\Pi_{j \neq k} \ell_j \right)^2 \tag{13}$$

Finally, the right-hand side of this equation is clearly equal to the sum of the areas of the orthogonal faces. This proves the theorem.

3 Closing comments

I hope that the above paragraphs will motivate the reader to think of the Pythagorean theorem as a property of orthogonality not connected with any particularities of planar geometry. In the interest of intellectual honesty, I should mention that the restriction to tetrahedra or affine simplices is no more crucial! Although I have chosen not go into this aspect here, the fact remains that a Pythagorean theorem holds also for more general bodies in Euclidean n-space. The hypothesis of orthogonality of the faces at the vertex may be replaced by the reference to mutually orthogonal projections in the conclusion. Thus, the modified result states that the volume of a hypersurface (the generalization of the "opposing face" in the simplicial case) and the volumes of the projections of the surface in the direction of the elements of an orthogonal basis for the ambient Euclidean space (the analogs of the "orthogonal faces") are related by the quadratic Pythagorean identity. This was known to Monge and Hachette around 1800 ([2], 22.8, p. 533). Here, I will say only that it is rather straightforward to extend the simplicial version of the theorem to parellelipipeds, and that then the techniques of differential calculus apply. Perhaps the reader's curiosity has been sufficiently stimulated so as to prompt a personal investigation at this point.

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