CSCI2243-Review Problems for the Second Midterm

November 11, 2017

This exam will cover propositional logic, sets and functions, predicate logic, classification of binary relations, and various methods of proof. The sample questions are drawn from the last three of these topics; the first two were covered amply in the materials for the first midterm.

These problems are culled from past exams. One of them, Problem 6, involves some number theory, but only in a rather superficial way, and you should be able to understand it and do it without intensive study of the number theory topics from Chapter 7.

1 Predicate logic and relations

This problem concerns a set S of middle-school students, and a function

$$f : S \to \mathcal{P}(S),$$

where each for each student $s \in S$, $f(s)$ is the set of students whom s likes. For example,

$$f(\text{Steve}) = \{\text{Jane, Albert}\}$$

means that Steve likes exactly two people, Jane and Albert.

We can describe a property of this world in three ways: by a sentence of English, by a sentence of predicate logic containing a relation symbol $L(x, y)$ which means that x likes y, and by an equation or inequality involving sets and the function f.

In the table below, you are given a number of scenarios expressed in one or two of these languages, and your job is to fill in the blank spaces in the table, translating the condition into the other languages.

$f(\texttt{Joe}) \subseteq f(\texttt{Ed})$	Ed likes everyone that Joe likes	*(a)*
(b)	Tom likes Jane and no one else	$L(\texttt{Tom}, \texttt{Jane}) \wedge \forall y (L(\texttt{Tom}, y) \rightarrow y = \texttt{Jane})$
(c)	Nobody likes themself[1]	*(d)*
$\{s \in S : f(s) = S\} \neq \emptyset$	*(e)*	*(f)*

(g) Suppose the students in this middle school form three cliques. Every student likes themself, and all the other members of their clique, but no student likes any member of a different clique. What kind of relation is L? A preorder? an equivalence relation? a partial order? a total order? none of these?

[1] After years of being a grammar snob, I've gone over to the dark side and decided that it's ok to use the singular 'they' in writing, and avoid such awkward constructions as 'no one likes himself or herself'.

2 A Product Identity

(a) Compute the values of

$$\prod_{j=1}^{n}(1+\frac{1}{j})$$

for $n = 1, 2, 3, 4$.

(b) Based on your answer in *(a)*, conjecture a general formula for

$$\prod_{j=1}^{n}(1+\frac{1}{j}),$$

then prove it using mathematical induction.

3 Understanding predicate logic.

The directed graph below depicts the ‘likes’ relation among a group of four people; for instance, the arrow drawn from Jack to Jill means Jack likes Jill, and we write this as $L(\text{Joe}, \text{Jane})$. Translate the formulas below into English, and tell whether they are true in the model, with a *brief* explanation.

(a) $\exists x L(x, \text{Jill})$.

(b) $\forall x \exists y L(y, x)$.

(c) $\exists x \exists y \exists z (L(x, z) \wedge L(y, z) \wedge x \neq y)$.

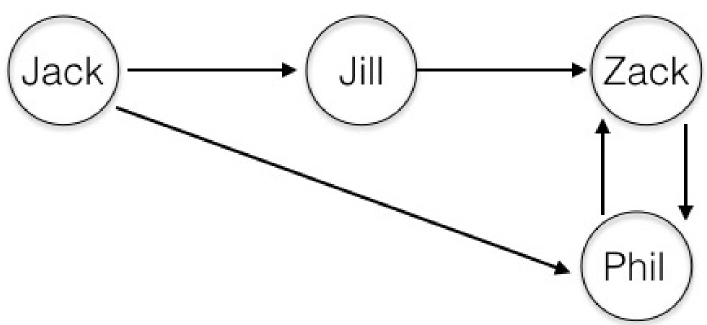

Figure 1:

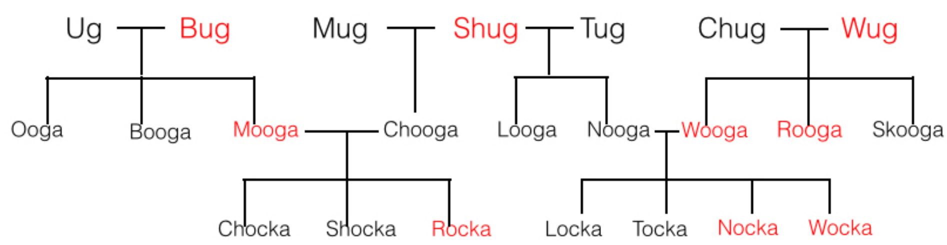

Figure 2: The Ug Family Tree.

4 Predicate Logic and Relations

Figure 2 shows the family tree of several generations of the Ug clan. Thus, for example, Chocka is the child of Mooga and Chooga.

Given such a family tree, containing a finite set X of individuals, we define a function

$$f : X \to \mathcal{P}(X),$$

where for each $x \in X$, $f(x)$ denotes the set of ancestors of x. We assume that each person is their own ancestor. Thus, for example,

$$f(\text{Chocka}) = \{\text{Chocka}, \text{Mooga.Chooga}, \text{Ug}, \text{Bug}, \text{Mug}, \text{Shug}\}.$$

(Note that Tug is *not* an ancestor of Chocka.)

We can describe a property of this family in three ways: by a sentence of English, by a sentence of predicate logic containing a relation symbol $A(x, y)$ which means that x is an ancestor of y, and by an equation or inequality involving sets and the function f.

In the table below, you are given a number of properties expressed in one or two of these languages, and your job is to fill in the blank spaces in the table, translating the condition into the other languages. The descriptions should work for any family tree with individuals having these names, not just the one pictured. (Observe that the property given in the third row of the table is not true in the pictured model.)

The expression in the bottom left cell may be a little tricky to figure out! The idea is that Chooga is a parent of Chocka if there is no one else in the chain of ancestors from Chocka to Chooga

Mug $\in f(\text{Shocka})$	Shocka is a descendant of Mug.	*(a)*
(b)	Wug has no ancestors other than himself	$\forall x(A(x, \text{Wug}) \to x = \text{Wug})$
(c)	Chocka and Tocka have an ancestor in common.	*(d)*
$\{x \in X : \text{Chooga} \in f(x)\} \cap f(\text{Chocka})$ $= \{\text{Chooga,Chocka}\}$	Chooga is a parent of Chocka.	*(e)*

(f) What kind of relation is A? An equivalence relation? A partial order? A preorder that is not a partial order? None of these?

5 Classifying relations; predicate logic.

The three directed graphs in Figure 3 define binary relations on the set of 7 vertices as follows: We write that $a \preceq b$ if $a = b$ or there is a *directed path* from a to b. (For example, in the last of the three drawings, if a is the vertex at the upper left and b the vertex at lower right, then $a \preceq b$, because there is such a path.) *In particular, all three relations are reflexive and transitive.*

(a) For each of the graphs, tell whether the relation defined is *(i)* an equivalence relation; *(ii)* a total order; *(iii)* a partial order that is not a total order; *(iv)* a preorder that is not a partial order.

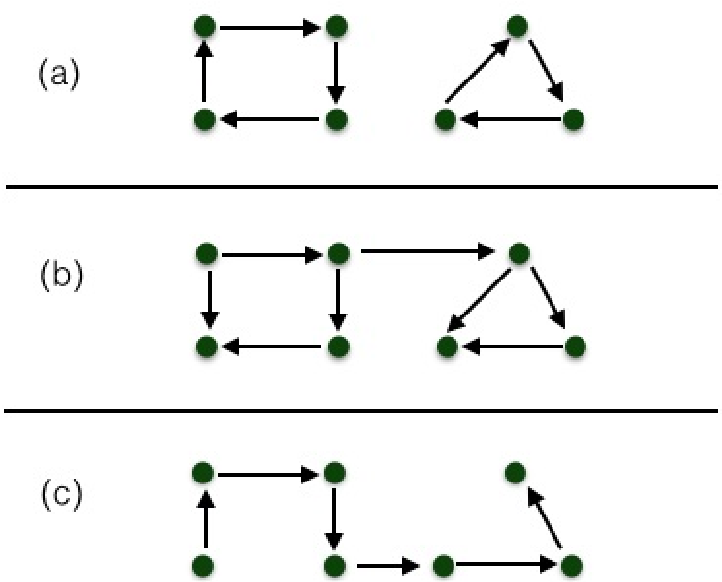

Figure 3: The three relations for Problem 2.

(b) For which of the graphs are the following sentences of predicate logic true? Consider each sentence in turn; it is possible that one of the sentences is true for a given relation while the other is false for the same relation. In addition, give a succinct translation into English of the last sentence.

(i)

$$\exists x \forall y (y \preceq x).$$

(ii)

$$\exists x \forall y ((x \preceq y) \rightarrow x = y).$$

(iii)

$$\exists x \exists y (\neg(x \preceq y) \wedge \neg(y \preceq x)).$$

6 Relations

Below we define four relations on the set $\mathbf{Z}^+$ of positive integers:

$$m\mathcal{R}_1 n \text{ iff } m|n.$$

$$m\mathcal{R}_2 n \text{ iff } \gcd(m,n) = 1.$$

$$m\mathcal{R}_3 n \text{ iff every prime that divides } m \text{ also divides } n.$$

$$m\mathcal{R}_4 n \text{ iff } m\mathcal{R}_3 n \text{ and } n\mathcal{R}_3 m.$$

(a) One of the four relations is an equivalence relation. Identify it, and determine the equivalence class that contains 12. (This equivalence class is infinite, but there is a succinct description.)

(b) One of the four relations is a preorder that is not a partial order. Identify it, and show with an example that it lacks a property required to make it a partial order.

(c) One of the four relations is a partial order that is not a total order. Identify it, and show with an example that it lacks a property required to make it a partial order.

(d) One of the four relations is not a preorder. Identify it, and show with an example that it lacks a property required to make it a preorder.

(e) Which of the relations satisfy the following sentences of predicate logic (so there is one answer, yes or no, for each formula and each relation)?

$$\exists x \forall y (x \mathcal{R} y)$$

$$\exists x \forall y (y \mathcal{R} x).$$

7 A Simple Proof

(a) Show that for any $n \geq 0$,

$$6|n^3 - n.$$

You can use any method you like—a proof by induction is possible, but not necessary.

(b) Write the statement above as a sentence of predicate logic interpreted in **N**. A complete solution to the problem will use the symbols 6, $+$, $=$ and $\times$, but not $|$, or $-$. However, you can get partial credit if your formula uses these other symbols.

8 A Summation Identity

(a) Compute the values of

$$\sum_{j=0}^{n} j(j+1)$$

for $n = 0, 1, 2, 3$.

(b) Prove by induction that

$$\sum_{j=0}^{n} j(j+1) = n(n+1)(n+2)/3$$

for all $n \geq 0$.