CSCI2243-Logic and Computation:Final Exam

December 12, 2015

1 p, q and r Have Lunch *(20 points)*

Perlmutter, Quimby and Rizzo eat lunch together every day at the same diner, which for some reason only serves tuna salad and chicken salad sandwiches. In this problem, you will model the three people's lunch orders in Propositional Logic, using variables p, q, r. We interpret p to mean 'Perlmutter orders tuna', and $\neg r$ to mean 'Rizzo orders chicken'. (A customer must order exactly one of these two items.)

(a) Write succinct, natural English interpretations for each of the two formulas

$$p \oplus q$$

and

$$\neg p \to r.$$

(Ideally, neither of your answers should use the word 'not'.)

Rizzo likes to order last, because she bases her decision on what her friends ordered. She uses the following criteria:

(i) If Perlmutter and Quimby order the same item, then Rizzo orders something different.

(ii) If Perlmutter and Quimby order different items, then Rizzo orders what Perlmutter ordered.

(b) Write formulas ϕ and ψ of Propositional Logic modeling each of these criteria. Your formulas should use as few connectives as possible, but you can make use of any of the connectives we introduced in class (*i.e.,* $\to$, $\leftrightarrow$, and $\oplus$ as well as $\wedge$, $\vee$, $\neg$.)

(c) Write a formula in disjunctive normal form that is equivalent to $\phi \wedge \psi$. You can use any method you want: algebraic manipulation, truth tables, just kind of talking your way through it, but be sure to give a careful account of your reasoning.

2 Relations *(30 points)*

Below we define four relations on the set $\mathbf{Z}^+$ of positive integers:

$$m\mathcal{R}_1 n \text{ iff } m|n.$$

$$m\mathcal{R}_2 n \text{ iff } \gcd(m,n) = 1.$$

$$m\mathcal{R}_3 n \text{ iff every prime that divides } m \text{ also divides } n.$$

$$m\mathcal{R}_4 n \text{ iff } m\mathcal{R}_3 n \text{ and } n\mathcal{R}_3 m.$$

(a) One of the four relations is an equivalence relation. Identify it, and determine the equivalence class that contains 12. (This equivalence class is infinite, but there is a succinct description.)

(b) One of the four relations is a preorder that is not a partial order. Identify it, and show with an example that it lacks a property required to make it a partial order.

(c) One of the four relations is a partial order that is not a total order. Identify it, and show with an example that it lacks a property required to make it a partial order.

(d) One of the four relations is not a preorder. Identify it, and show with an example that it lacks a property required to make it a preorder.

(e) Which of the relations satisfy the following sentences of predicate logic (so there is one answer, yes or no, for each formula and each relation)?

$$\exists x \forall y (x\mathcal{R}y)$$

$$\exists x \forall y (y\mathcal{R}x).$$

3 Sets, Functions, Cardinality *(25 points)*

Let X be the set of nonempty subsets of $\{1, 2, 3, 4, 5\}$. That is,

$$X = \{A \in \mathcal{P}(\{1, 2, 3, 4, 5\} : A \neq \emptyset\}.$$

(a) What is $|X|$? Give the exact numerical answer.

(b) What is $|Y|$, where $Y = \{A \in X : |A| = 2\}$? Give the exact numerical answer.

(c) What is $|W|$, where $W = \{f : \{1, 2, 3, 4, 5\} \to Y\}$? (That is, W is the set of functions from $\{1, 2, 3, 4, 5\}$ to Y. You can give the answer as an unevaluated expression involving addition, multiplication, and exponentiation, but actually it is quite easy to caculate the exact answer.)

(d) Does the set W defined above include any onto functions? does it include any one-to-one functions? Justify your answer.

(e) Now let $\mathbf{N}$, as usual, denote the set of natural numbers, and consider the sets:

$$V = \{A \subseteq \mathbf{N} : |A| = 2\}$$

and

$$U = \{f : \mathbf{N} \to V\}.$$

Does U contain any one-to-one functions? Does it contain any onto functions?

4 A Simple Proof *(20 points)*

(a) Show that for any $n \geq 0$,

$$6|n^3 - n.$$

You can use any method you like—a proof by induction is possible, but not necessary.

(b) Write the statement above as a sentence of predicate logic interpreted in $\mathbf{N}$. A complete solution to the problem will use the symbols 6, $+$, $=$ and $\times$, but not $|$, or $-$. However, you can get partial credit if your formula uses these other symbols.

5 A Summation Identity *(20 points)*

(a) Compute the values of

$$\sum_{j=0}^{n} j(j+1)$$

for $n = 0, 1, 2, 3$.

(b) Prove by induction that

$$\sum_{j=0}^{n} j(j+1) = n(n+1)(n+2)/3$$

for all $n \geq 0$.

6 Some Questions about Integers *(25 points)*

(a) Find the base ten representation of the integer whose base eight representation is 123.

(b) Find the base eight representation of the number whose hexadecimal representation is AB0C.

(c) Find

$$33\cdots3_8 \bmod 7,$$

where there are one hundred 3's in the base 8 representation.

(d) Find $2^{3600} \bmod 15$.

(e) Find $28^{3600} \bmod 37$.

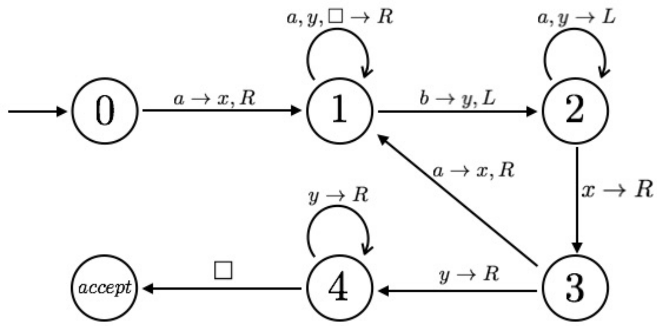

Figure 1: Turing machine for Problem 8

7 Regular Expressions and Automata *(12 points)*

(a) Find all words of length 3 or less in the language represented by the regular expression $(a + ab)^*$.

(b) Give a succinct description in English of this language.

(c) Draw the state diagram of a DFA that recognizes this language. There is a solution with three states, two of which are accepting states.

8 Turing Machines *(13 points)*

The accompanying figure is the state diagram of a Turing machine $\mathcal{M}$ that accepts all the strings of the form $a^k b^k$, where $k > 0$, and no others. The labeling follows our usual conventions for such diagrams: We only indicate a symbol to write if it is different from the symbol scanned, and any transition that is missing is assumed to lead to the reject state.

(a) Trace the execution of this machine on the strings ab and aab. Each step of the execution should show the complete configuration of the machine: tape contents, state, and position of the read/write head. If you find that the simulation is running more that ten steps, you should stop.

(b) Let L be the set of all strings of the form $a^k b^k$, where $k > 0$. Does $\mathcal{M}$ recognize L? Does $\mathcal{M}$ decide L? Is L Turing-recognizable? Is it Turing-decidable? (So there are four questions here.)

HAVE A GREAT BREAK