CSCI2243-Assignment 3

Assigned Thursday, September 14, due Friday, September 22 at 11PM

Read the first two sections of Chapter 3. This week's problems are straight from that Chapter. Don't panic at the large number of them, most are very short.

Give complete answers to the questions—for problems 1 and 2, write the statement out before you write 'True' or 'False'. For Problem 5(a), write $A \cup B =$ before you give the solution. (And use correct notation for the solution!)

Problems: 1,2,4,5,6,7(c),8,9,10(a,b,c,g), 11, 12(a,c,d), 13(a,c,d).

A note on the proofs. Problems 12 and 13 ask you to prove something—you either have to prove that the statement is true in general, or find a single counterexample. Here, to guide you, are solutions to parts (b) and (e) of Problem 13. You can see that the proof for 13(e) follows a standard pattern for proving that two sets are equal, and is quite mechanical. If you tried to repeat this pattern for part (b), you should get to a point where you say 'uh-oh, that...isn't always true', and this should lead you to a counterexample.

13(e) This is true.

A proof that two sets A, B are equal usually proceeds by showing that $A \subseteq B$ and $B \subseteq A$. So the pattern in these arguments is

> Suppose $x \in A$. Blah-blah. So $x \in B$.
>
> Conversely, suppose $x \in B$. Yadda-yadda. So $x \in A$.
>
> Thus $A = B$.

Here then, is the proof of 13(e), with the blah-blah and yadda-yadda filled in.

If $x \in f^{-1}(Z_1 \cap Z_2)$, then $f(x) \in Z_1 \cap Z_2$. This means $f(x) \in Z_1$ and $f(x) \in Z_2$. Thus $x \in f^{-1}(Z_1)$ and $x \in f^{-1}(Z_2)$. That means $x \in f^{-1}(Z_1) \cap f^{-1}(Z_2)$. This shows one direction of inclusion, namely $f^{-1}(Z_1 \cap Z_2) \subseteq f^{-1}(Z_1) \cap f^{-1}(Z_2)$.

Now for the opposite inclusion: If $x \in f^{-1}(Z_1) \cap f^{-1}(Z_2)$, then $x \in f^{-1}(Z_1)$, and $x \in f^{-1}(Z_2)$. So $f(x) \in Z_1$ and $f(x) \in Z_2$. This means $f(x) \in Z_1 \cap Z_2$, so $x \in f^{-1}(Z_1 \cap Z_2)$. This shows $f^{-1}(Z_1) \cap f^{-1}(Z_2) \subseteq f^{-1}(Z_1 \cap Z_2)$. Since we have shown inclusions in both directions, the two sides are equal.

13(b) This is false.

Let's see what would happen if we tried to prove it, following the pattern above. We might first try to show the inclusion:

$$f(W_1 \cap W_2) \subseteq f(W_1) \cap f(W_2).$$

So suppose $y \in f(W_1 \cap W_2)$. This means $y = f(x)$ for some $x \in W_1 \cap W_2$. Thus in particular, $x \in W_1$, so $y = f(x) \in f(W_1)$. Exactly the same reasoning shows $y \in f(W_2)$. Thus $y \in f(W_1) \cap f(W_2)$. So far, so good. Now let's try to show the opposite inclusion

$$f(W_1) \cap f(W_2) \subseteq f(W_1 \cap W_2).$$

Let us suppose $y \in f(W_1) \cap f(W_2)$. Then $y = f(x_1)$ for some $x_1 \in W_1$, and also $y = f(x_2)$ for some $x_2 \in W_2$. But we have to show $y = f(x)$ for some $x \in W_1 \cap W_2$. This would be fine if x_1 and x_2 were the same element, but there is no guarantee of that. We're stuck.

The fact that the proof hit a wall at this point helps us construct an extremely simple counterexample, which proves that the proposed equation is false. Let $X = \{1, 2\}$ and let $Y = \{1\}$. We define $f : X \to Y$ as the only thing a function from X to Y can be in this case: $f(1) = f(2) = 1$. Let $W_1 = \{1\}$ and $W_2 = \{2\}$. Then $W_1 \cap W_2 = \emptyset$, so $f(W_1 \cap W_2) = f(\emptyset) = \emptyset$. But $f(W_1) = f(W_2) = \{1\}$, so $f(W_1) \cap f(W_2) = \{1\}$, and thus in this case $f(W_1 \cap W_2) \neq f(W_1) \cap f(W_2)$.