CSCI2243-First exam, with solutions.

CSCI2243-Logic and Computation

There were three versions of Problem 1. The solutions displayed below illustrate the range of methods available.

1. Consider the two propositional formulas

$$p \to (p \to q), (p \to p) \to q$$

To solve this problem, you may use any method: algebra, truth tables, talking it through; but you must justify your answer carefully.

(a) For each of the two formulas, tell whether it is a tautology, and tell whether it is satisfiable.

(b) For each of the two formulas, find an equivalent formula that is as simple as possible, and that uses no connectives other than $\neg$ and $\vee$.

Solution. Let's attack this one with algebra, since it is easy to rewrite $\to$ in terms of $\vee$ and $\neg$:

$$\begin{aligned} p \to (p \to q) &\equiv \neg p \vee (\neg p \vee q) \\ &\equiv (\neg p \vee \neg p) \vee q) \\ &\equiv \neg p \vee q. \end{aligned}$$

$$\begin{aligned} (p \to p) \to q &\equiv (\neg p \vee p) \to q \\ &\equiv \mathbf{T} \to q \\ &\equiv \mathbf{F} \vee q \\ &\equiv q. \end{aligned}$$

This is the answer to part (b). We can also see by inspecting the formulas that neither is a tautology (for example p true and q false makes them both false) but both are satisfiable (for example, any assignment with q true makes them both true). 1. Consider the two propositional formulas

$$(p \to q) \oplus p, p \to (q \oplus p)$$

To solve this problem, you may use any method: algebra, truth tables, talking it through; but you must justify your answer carefully.

(a) For each of the two formulas, tell whether it is a tautology, and tell whether it is satisfiable.

(b) For each of the two formulas, find an equivalent formula that is as simple as possible, and that uses no connectives other than $\neg$ and $\vee$.

Solution. Let's use a truth table for this one.

p	q	$p \to p$	$(p \to p) \oplus q$	$p \oplus q$	$p \to (p \oplus q)$
T	T	T	**F**	F	**F**
T	F	T	**T**	T	**T**
F	T	T	**T**	T	**T**
F	F	T	**T**	F	**T**

The two formulas give the same result—they are true as long as p and q are not both true. In other words, they are equivalent to

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

Again, we have both satisfying and nonsatisfying assignments, so both formulas are satisfiable, neither is a tautology.

1. Consider the two propositional formulas

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To solve this problem, you may use any method: algebra, truth tables, talking it through; but you must justify your answer carefully.

(a) For each of the two formulas, tell whether it is a tautology, and tell whether it is satisfiable.

(b) For each of the two formulas, find an equivalent formula that is as simple as possible, and that uses no connectives other than $\neg$ and $\vee$. **Solution.** Let's talk this

one through: Since $p \to p$ is always true, the only way $(p \to p) \oplus q$ can be false is for q to be true. Thus this formula is true whenever q is false, so it is equivalent to $\neg q$. $p \to (p \oplus q)$ is false if p is true and $p \oplus q$ is false; that is, if p and q are both true. So this is equivalent to $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (the same result we found with a truth table in the previous version of this problem). Once again, both formulas have satisfying and nonsatisfying assignments, so they are both satisfiable, neither is a tautology.

2. At Taquería El Pelón, you can get a burrito with either a delicious filling of carnitas (c), or a meatless version that has only rice, beans and cheese. You can add guacamole (g), sour cream (s), or both, to your burrito, or you can skip the extras. Thus there are eight possible combinations for your burrito order, but you have your own restrictions, namely:

> When you order the no-meat version, you get both of the extras.
>
> When you order the carnitas filling, you get exactly one of the extras (*i.e.,* either guacamole or sour cream, but not both).

Naturally, you will use a boolean satisfiability solver to generate a lunch order that satisfies these requirements. You first have to encode your requirements in CNF. Which of the following three possibilities could not possibly be a clause in the CNF? (A clause is impossible if it *overly* constrains by eliminating legal solutions. There are one or more such clauses in the list below.) Be sure to justify your answer.

$$c \vee g$$

$$\neg c \vee g$$

$$\neg c \vee g \vee s.$$

Solution. A clause cannot occur if it rules out a correct solution. Another way to say this is that a clause can occur if it is true in every solution. $c \vee g$ says you either get carnitas or guacamole, which is indeed true in every solution, because if you don't get carnitas, then you have to get guacamole, along with sour cream. $\neg c \vee g \vee s$ is also true in every solution: in fact $g \vee s$ is true in every solution, because you always get at least one side, regardless of whether you get the carnitas. $\neg c \vee g$ is the culprit here: It is false if you order carnitas and sour cream without guacamole, although this order is permitted by the rules.

2. At Taquería El Pelón, you can get a burrito with either a delicious filling of carnitas (c), or a meatless version that has only rice, beans and cheese. You can add guacamole (g), sour cream (s), or both, to your burrito, or you can skip the extras. Thus there are eight possible combinations for your burrito order, but you have your own restrictions, namely:

> When you order the no-meat version, you get both of the extras.
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> When you order the carnitas filling, you get exactly one of the extras (*i.e.,* either guacamole or sour cream, but not both).

Needless to say, you will design and build an electronic digital circuit to determine if your lunch order meets these requirements. Two candidate circuits are shown below, but only one of them is correct. Tell which is the incorrect one, and explain why. (Note that the output is supposed to be 1 if the lunch order meets the requirements. The labels a, b, c, x in the diagrams were generated automatically by the software, and should be ignored.)

There are two ways to approach this: Since you are told that exactly one of the two is correct, you can either try to identify the correct one or the incorrect one. For the former, note that the bottom diagram is just a drawing in circuit form of the complete DNF

$$(\neg c \wedge g \wedge s) \vee (c \wedge \neg g \wedge s) \vee (c \wedge g \wedge \neg s),$$

which gives exactly the three satisfying assignments to this problem.

For the latter, you might wonder if there is something suspicious about the two AND gates in the top diagram that have only two inputs. If you translate the circuit into DNF, you see that the disjunct corresponding to the top AND gate is

$$\neg c \wedge g.$$

This means that *any* order with guacamole and no carnitas will be accepted, even though guacamole, no carnitas, and sour cream is not allowed by the conditions.

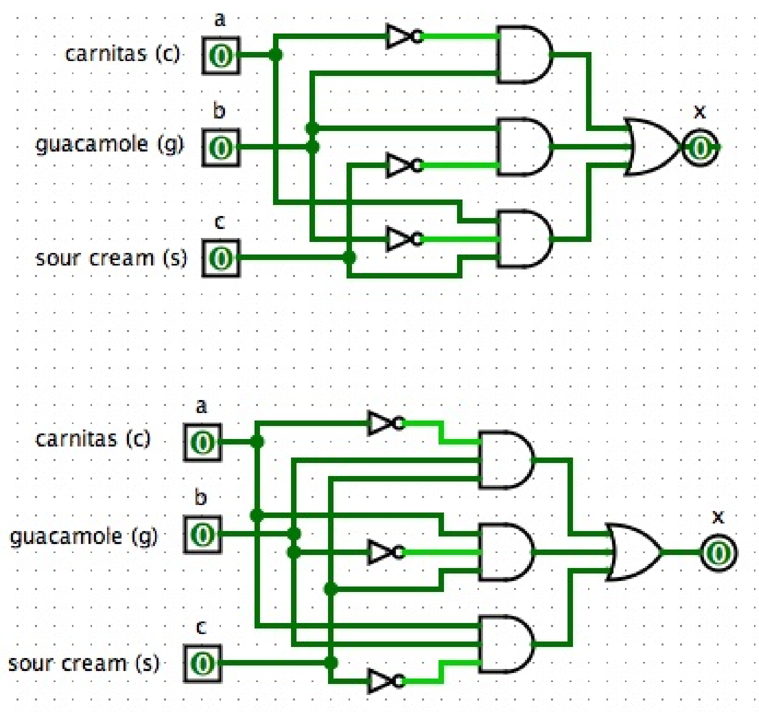

Figure 1: *Circuits for Problem 2*

So the top circuit is incorrect. (You could also look at the second AND gate, which permits any order with guacamole and no sour cream, also an error).

3. All the parts of this problem refer to the following sets:

$$A = \{1, 2, 3, 4, 5, 6\}$$

$$B = \{5, 6, 7, 8\}.$$

(a) Determine

$$|(A \cap B) \cup (A \backslash B)|.$$

Solution. You can tabulate it carefully, but you might also notice that $(A \cap B) \cup (A \backslash B) = A$ for any sets A, B, so the answer here is $|A| = 6$.

(b) List all the elements of

$$\{x \in \mathcal{P}(A \cap B) : |x| = 1\}.$$

(Be careful how you write these!

Solution.

$$\mathcal{P}(A \cap B) = \{\emptyset, \{5\}, \{6\}, \{5, 6\}\},$$

so the elements in question are $\{5\}, \{6\}$.

(c) Consider the function

$$f : A \times B \to A$$

given by

$$f((a, b)) = \min(a, b)$$

for all $(a, b) \in A \times B$. For example, $f((4, 7)) = 4$. Is f one-to-one? Is f onto? Justify your answer.

Solution. $f(4, 7) = 4 = f(4, 8)$, which shows you that the function is not one-to-one. Alternatively, you can observe that in this case, $|A \times B| > |A|$ so that *no* function from $A \times B$ to A can be one-to-one.

Since $f(a, 8) = a$ for every $a \in A$, we conclude that every element of A is in the range of f, so f is onto. As mentioned in class, it is not sufficient to note that $|A \times B| \geq |A|$, because this does not by itself guarantee that a function from $A \times B$ to A is onto.

3. All the parts of this problem refer to the following sets:

$$A = \{1, 2, 3, 4, 5, 6\}$$

$$B = \{5, 6, 7, 8\}.$$

(a) List all the elements of

$$(A \backslash B) \cup (B \backslash A).$$

Solution.

$$(A \backslash B) \cup (B \backslash A) = \{1, 2, 3, 4, 7, 8\}.$$

(This is the symmetric difference $A \Delta B$.)
(b) List all the elements of

$$\{x \in \mathcal{P}(A \cap B) : |x| < 2\}.$$

(Be careful how you write these!

Solution. The elements are $\emptyset, \{5\}, \{6\}$. (See above for an explanation.)

(c) Consider the function

$$f : A \times B \to B$$

given by

$$f((a, b)) = \max(a, b)$$

for all $(a, b) \in A \times B$. For example, $f((4, 7)) = 7$. Is f one-to-one? Is f onto? Justify your answer.

Solution. We have $f(4, 7) = f(3, 7) = 7$, so f is not one-to-one. We have $f(1, b) = b$ for all $b \in B$, so f is onto. (See the solution to the other version of this problem for comments, which apply here as well.)

4. Is there a one-to-one function

$$f : \mathbf{Q} \times \mathbf{Q} \to \mathbf{R}?$$

Is there an onto function with this domain and range? Justify your answers—you may cite any theorem we studied in the course.

Solution. The magic words are *countable* and *uncountable*. We saw in class that $\mathbf{R}$ is an uncountable set, while $\mathbf{Q}$ is countable. We also saw how to take the cartesian product of two countable sets and put it in one-to-one correspondence with the set of positive integers, so $\mathbf{Q} \times \mathbf{Q}$ is also countable. Short answer then: we have a domain X and codomain Y where Y has strictly more elements than X. So just as with finite sets, there can be a one-to-one function, but there cannot be an onto function.

For more detail as to why this is (you didn't need to write this): $\mathbf{Q} \times \mathbf{Q}$ countable means there is a bijection $g : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Z}^+$, but $\mathbf{Z}^+ \subseteq \mathbf{R}$, so this gives a one-to-one function into **R**. If there were an onto function

$$f : \mathbf{Q} \times \mathbf{Q} \to \mathbf{R}$$

then $h = f \circ g^{-1} : \mathbf{Z}^+ \to \mathbf{R}$ would be onto, and the values

$$h(1), h(2), h(3), \ldots,$$

would be a list containing all the elements of **R**, contradicting uncountability.

4. Let $W = \{a, b, c, \ldots, z\}^*$ denote the set of all finite strings over the alphabet $\{a, b, c, \ldots, z\}$. For instance, W contains elements like

```
cat
dog
qkwjfklqnxmnvdjnjaknvnknapqienvjmrxgnuf
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Is there a one-to-one function

$$f : \mathbf{R} \to W?$$

Is there an onto function with this domain and range? Justify your answers—you may cite any theorem we studied in the course.

Solution. W is countable and **R** is uncountable (from class), **R** has strictly more elements than W, and thus there is an onto function but no one-to-one function.

For more detail, there is a bijection $g : \mathbf{Z}^+ \to W$. If we extend the domain of g so that it includes all of **R**, say by defining $g(a) = \texttt{'cat'}$ for any $a \in \mathbf{R} \backslash \mathbf{Z}^+$ (this is totally arbitrary) then we get an onto function. If there were a one-to-one function f, then $g^{-1} \circ f$ would be a one-to-one function from $\mathbf{R}$ into $\mathbf{Z}^+$, which is a bijection between **R** and a subset of $\mathbf{Z}^+$. This would make **R** countable, contradicting what we know.