Second exam-with solutions

CSCI2243-Logic and Computation

November 17, 2017

1 Problems and Solutions

1. Figure 1 shows the prerequisite structure diagram for required courses in the Computer Science major, copied from the departmental website. The dashed black double arrow between Computer Organization and Logic and Computation is something that I added and that **you should ignore in completing parts (a) and (b)**–we will return to it in part (c).

(a) We define a relation $\preceq$ on the set of courses by $A \preceq B$ if either $A = B$ or A must be completed before B is begun. This is precisely the case if there is a directed path of length at least 0 from A to B. What kind of relation is $\preceq$? A total order? partial order? preorder? Give more than one answer if more than one of these categories is applicable.

Solution. Whenever we have a relation defined this way on the vertices of a directed graph ($a \preceq b$ if and only if there is a directed path of length ≥ 0 from a to b) the relation is automatically reflexive and transitive, and thus a *preorder*. This one is also a *partial order* because course A cannot be a prerequisite for course B and have B as a prerequisite: in the directed graph setting, this means that there are no directed cycles in the graph, as is the case here. It is not a total order, because, for example, Computer Organization and Randomness are not comparable: neither is a prereq for the other (many other pairs illustrate this).

(b) Let us define $A \sim B$ if A and B can be taken during the same semester. (This question is about the rules, not what real students do—for example some students *do* take Algorithms during the same semester they take this course, but they're not supposed to!) The symbol $\sim$ was used to suggest this might be an equivalence relation, but in fact it is NOT an equivalence relation. Which property

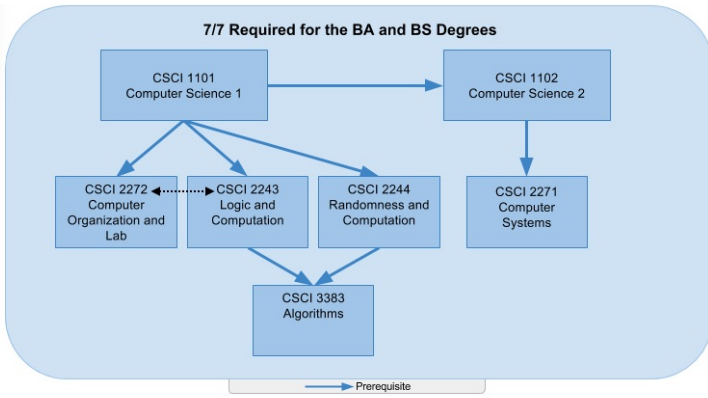

Figure 1: Prerequisite chart for CS major requirements.

or properties of equivalence relations does $\sim$ fail to have? Justify your claim with examples.

Solution. 'A and B can be taken in the same semester' is obviously both reflexive and symmetric, so if this fails to be an equivalence relation, it must be because transitivity fails. To see this, note that Randomess and Systems can be taken during the same semester, and both Systems and Algorithms can be taken during the same semester, but Randomness cannot be taken in the same semester as Algorithms—thus this relation is not transitive.

Some students wrote that this could not be a reflexive relation, because 'it makes no sense to take Algorithms and Algorithms in the same semester, because you cannot take a course twice in the same semester'. I gave a couple of points for this odd (and particularly unmathematical) interpretation. Some students even insisted, in spite of what the problem says, that this IS an equivalence relation.

(c) The dashed double arrow between CSCI2243 and CSCI2272 is meant to indicate *mutual corequisites*–courses that must be taken at the same time. (No, we do not have such a requirement in our department, but imagine that we do.) We redefine $A \preceq B$ to mean that A cannot be taken later than B. Answer the question in part (a) for this new definition of $\preceq$. What, if anything, has changed?

Solution. This is still reflexive and transitive as in (a), and thus $\preceq$ is still a preorder. But the new arrows destroy the antisymmetry: Computer Organization cannot be taken later than Logic, and Logic cannot be taken later than Computer Organization. (Put differently, the arrows between these two courses give a directed cycle in the graph.)

2. The following sentences of predicate logic are statements about the arithmetic and order of numbers. Whether they are true or not depends on what set of numbers we are using as the universe. For each sentence, give a brief restatement in informal English, and then identify in which of the universes $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ the sentence is true. If the sentence is *false* in an interpretation, give an example that demonstrates this.

(a) $\exists x \forall y (x \leq y)$.

Solution. This says there is a least element. That is true in $\mathbf{N}$, where 0 is the least element, but false in the other domains, because we can always subtract 1 from any number to get a smaller one.

(b) $\forall x \exists y (x \leq y)$

Solution. This is actually true in *any* nonempty set of numbers, since given any x we can pick $y = x$ and thus have $x \leq y$. It is ok if you read the $\leq$ as $<$ and interpreted the sentence to mean that there is no largest number. This is also true in all the domains, because given any x, we can take $y = z + 1$.

(c) $\forall x (x \geq 0 \to \exists y (y \times y = x))$.

Solution. This says that every nonnegative element has a square root in the domain. This is false in $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$, because $\sqrt{2}$ is irrational. It is true in $\mathbf{R}$, which include $\sqrt{2}$ as an element.

(d) $\forall x \exists y (x < y \wedge \forall z (z \leq x \vee z \geq y))$.

Solution. This says that for every element x there is a next largest element y. This is true in $\mathbf{N}$ and $\mathbf{Z}$, because we can pick $y = x + 1$. It is false in $\mathbf{Q}$ and $\mathbf{R}$ because given any two elements $x < y$, we can find an element (for example, their average $(x + y)/2$) that is strictly between them.

Note that this one is certainly harder than the others, because it has three levels of alternating quantifiers instead of two.

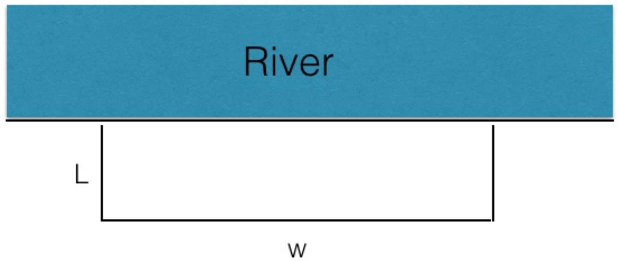

Figure 2: A tiresome old math problem about a farmer and a fence.

3. They always give this problem in Intro Calculus courses, which I've always thought is kind of silly, because there is a very simple proof of the result of this problem, as you will now show.[1]

A chicken farmer wants to build a rectangular pen for her hens. One side of the rectangle is the bank of a very straight river on her property. She has 40 feet of fencing to make the other three sides, so that

$$2L + w = 40.$$

Prove that the area of the pen will maximized if it is half a square, that is, if $L = 10, w = 20$. No calculus allowed–you just have to state and prove the relevant inequality. Make sure that both the statement and the proof are carefully presented, with all the relevant hypotheses and the steps in the right order. [2]

Solution. Statement: Let $L, w \in \mathbf{R}$ such that $2L + w = 40$. Then $Lw \leq 200$. Proof: Since $2L + w = 40$, $w = 40 - 2L$, and thus

$$Lw = L(40 - 2L) = 40L - 2L^2.$$

We can complete the proof using an argument by contradiction. Suppose, contrary to what we are trying to prove, that

$$40L - 2L^2 > 200.$$

[1] In fairness to the Calculus courses, it *is* a pretty good demonstration of Calculus principles, even if that is overkill for this problem.

[2] I do not know why she wants to build a rectangular pen when a semicircular pen would enclose the largest area. I also do not know if chickens can swim. It's a dumb math problem, not real chicken farming.

Then

$$0 > 200 - 40L + 2L^2 = 2(L - 10)^2.$$

We can divide both sides of the inequality by 2 and conclude that $(L - 10)^2 < 0$, which is a contradiction because squares of real numbers are never negative. (Alternatively, you could begin the proof with true statement $(L - 10)^2 \geq 0$ and derive the result $Lw \leq 200$. In either case the trick is that somehow or other you reduce the problem to the fact that squares are never negative.)

This problem caused the most trouble for students, because it was not drawn from the cookbook . My thought, incidentally, was that you would quickly see that the goal is to prove that $L(40 - 2L) \leq 200$ for all L, and remember that a common way of proving such inequalities was to find a square lurking in there somewhere.

A common error is 'if you have a hammer, everything looks like a nail' thinking: In this case, the hammer is mathematical induction. Induction is perfectly useless here, as there is no integer parameter to induct on. It does not make a great deal of sense to try to prove the statement only for integer values of L and w, since we would then not rule out the possibility that the maximum area occurs when the dimensions are not whole numbers.

4. (a) Compute the values of

$$\sum_{j=1}^{n} j \cdot 2^{j-1}$$

for $n = 1, 2, 3, 4$.

Solution. The values of the successive summands for $n = 1, 2, 3, 4$ are

$$1 \cdot 2^0 = 1, 2 \cdot 2^1 = 4, 3 \cdot 2^2 = 12, 4 \cdot 2^3 = 32.$$

Thus the successive values of the sum are

$$1, 1 + 4 = 5, 1 + 4 + 12 = 17, 1 + 4 + 12 + 32 = 49.$$

Of course, you can 'cheat' on this problem and just use the formula that is given to you in part (b)!

(b) Prove that

$$\sum_{j=1}^{n} j \cdot 2^{j-1} = (n-1) \cdot 2^n + 1$$

for all $n \in \mathbf{Z}^+$. The logical format of the proof is quite as important as getting the (relatively simple) algebra correct.

Solution. We use a proof by induction. For the base case, when $n = 1$, our tabulation above shows that the sum has value 1, and the formula gives $0 \cdot 2 + 1 = 1$, so the statement is true in this case. For the inductive step, suppose $n \geq 1$ and

$$\sum_{j=1}^{n} j \cdot 2^{j-1} = (n-1) \cdot 2^n + 1.$$

We need to show

$$\sum_{j=1}^{n+1} j \cdot 2^{j-1} = n \cdot 2^{n+1} + 1.$$

We have

$$\begin{aligned}
\sum_{j=1}^{n+1} j \cdot 2^{j-1} &= \Big(\sum_{j=1}^{n} j \cdot 2^{j-1}\Big) + (n+1)2^n \\
&= (n-1) \cdot 2^n + 1 + (n+1)2^n (by the inductive hypothesis) \\
&= (n - 1 + n + 1) \cdot 2^n + 1 \\
&= 2n \cdot 2^n + 1 \\
&= n \cdot 2^{n+1} + 1.
\end{aligned}$$

Presentation was an important part of the grade here. It was not enough to write all the correct calculations; you need to write out what you are assuming, what you intend to prove, and where you apply your assumption.

2 Results

As disappointing as the results of the first exam were gratifying. So much so that I adopted a feel-good grading policy and based your score on the two problems on which you received the highest scores (I all but threw out problem 3, which was done correctly by a only a tiny handful of students). With this modification, the grades were as follows: Maximum score, 36. Top quartile, 33 and above. Median 30. Bottom quartile, 26 and below. Mean, 28.8. Of 96 students taking the exam, 15 got 'perfect' scores (the result of this odd grading method), 11 received scores of 20 or below.