A collection of problems from earlier midterm exams

CSCI2243-Logic and Computation

1.*Basic syntax and semantics of propositional formulas.* Construct the truth table for the formula displayed below. Then answer the three questions following it.

$$(p \to q) \oplus \neg(p \vee q).$$

(a) Is the formula a *tautology?* What is it about the truth table that tells you this?

(b) Is the formula *satisfiable?* What is it about the truth table that tells you this?

(c) Find an *equivalent* formula that is as simple as possible.

2. *Modeling with propositional formulas.* Twins Penelope and Quentin need to find seats at the dinner table, which has places for three people. Since Penelope is right-handed and Quentin is left-handed, **Quentin cannot sit immediately to the right of Penelope.** Since Penelope has very sensitive eyes, and seat 1 is opposite a window, **Penelope cannot sit in seat 1.** Note that the diagram shows the table from above, so that seat 2 is to the **right** of seat 1.

Imagine giving the problem of determining an acceptable seating arrangement to a Satisfiability Solver. There are six boolean variables $p_1, p_2, p_3, q_1, q_2, q_3$ meaning 'Penelope sits in seat 1, *etc.*. The CNF specification of this simple problem has quite a few clauses.

(a) One of the clauses is

$$\neg p_1 \vee \neg q_1$$

and another is

$$\neg p_1 \vee \neg p_2.$$

Explain in English what these clauses say, and why they are part of the specification.

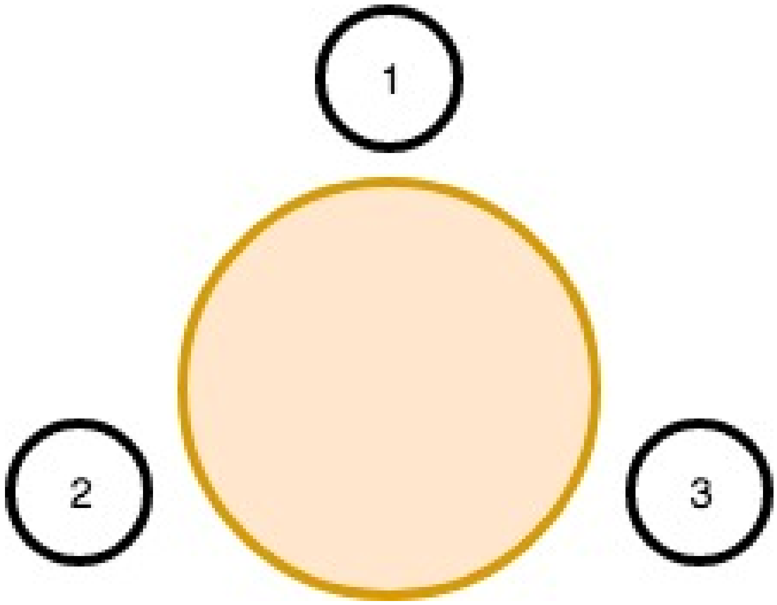

Figure 1: *Dinner table for Problem 2.*

(b) Write a clause or clauses that express the first constraint; namely that Quentin cannot sit immediately to the right of Penelope.

(c) Of course, you do not need a computer program to solve this problem. Write the truth values assigned to the variables $p_i, q_i,\ i = 1, 2, 3,$ for *one* satisfying assignment. How many satisfying assignments are there altogether?

4. *Sets and Functions* This problem refers to the sets and functions defined below. Each part of the question asks you either to identify a set, in which case you should list the elements, or a number.

$$A = \{0, 1, 2, 3\}$$

$$B = \{0, 2, 4, 6\}.$$

$$f : A \to B,$$

where $f(a) = 2|a - 2|$ for all $a \in A$.

$$g : B \to A$$

where $g(b) = b/2$ for all $a \in A$.

(a) $(B - A) \cup (A - B)$.

(b) $|\mathcal{P}(A \cap B)|$.

(c) $|\text{range}(g)|$. What does this tell you about whether g is one-to-one or onto?

(d) $g{\circ}f\,(3) + f{\circ}g\,(2)$.

5. *Basic syntax and semantics of propositional formulas.* Consider the formula

$$(p \oplus q) \to (\neg p \wedge q).$$

(a) Write an equivalent formula in disjunctive normal form. You must justify your result: you may do this by means of a truth table, by algebraic manipulation of the formula, or by talking it through.

(b) Find both a satisfying assignment and a non-satisfying assignment for this formula.

(c) Find an equivalent formula that uses only the connectives $\wedge, \vee, \neg$ and is as simple as possible (uses as few symbols as possible).

(d) Let us interpret p as 'Emily orders soup' and q as 'Emily orders salad'. This cafeteria has no rules: You can order both soup and salad if you like; you can even have a sandwich. The following are two possible interpretations in English of our formula. Which of them are correct? (The answer could be 'both' or 'neither'.)

(i) Whenever Emily chooses between soup and salad (that is, whenever she has one but not both), she orders the salad and not the soup.

(ii) Emily never orders soup unless she gets salad, too.

6. *Sets*

(a) Let $A = \{a, b, c\}$, $B = \{b, c, d\}$. List all the elements of $(A \cap B) \times (A \cup B)$.

(b) Let A and B be arbitrary sets. Which of the following are true? Justify briefly.

$$A \cup B = (A - B) \cup B.$$

$$A \cup B = (A - B) \cup (B - A).$$

(c) List all the elements of $\mathcal{P}(\mathcal{P}(\{1\}))$.

7. *Functions.*

(a) Figure 2 illustrates several assignments of elements of a set A to elements of A. Not all of these assignments define a function $f : A \to A$. Tell which of them are functions, and, in the cases where they are functions, whether these functions are one-to-one, onto, both, or neither.

(b) Let A and B be arbitrary sets. If $f : A \to B$ is one-to one and $g : B \to A$ is onto, if $g \circ f$ necessarily onto? The answer is no. Show this by giving a pictorial example, in the style of the diagrams in Figure 2.

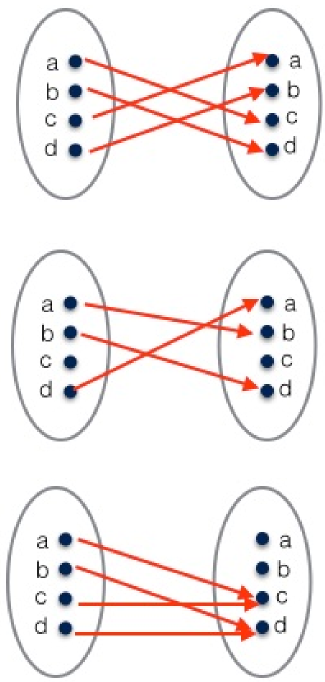

Figure 2: The three relations for Problem 7.

(c) Let $A = \mathbf{Z}^+$, and let

$$B = \{x \in \mathbf{R} : 0 \leq x \leq 1\}.$$

Is there a one-to-one function $f : A \to B$? Is there an onto function $f : A \to B$? You may use any theorem you learned in the class to justify your answer.

8. *Propositional Logic.*

(a) Write a sentence of propositional logic in *conjunctive normal form* that is equivalent to

$$p \to (q \wedge r).$$

(b) How many satisfying assignments does the formula in (a) have? Explain briefly.

(c) Find a formula that is as simple as possible (uses as few symbols as

possible) that is equivalent to

$$(\neg p \vee q) \leftrightarrow p.$$

You may use any method you like—algebra, truth tables, or just talking it through, but you must have some explanation of your reasoning.

9. *Sets*

(a) Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{4, 5, 6, 7, 8, 9, 10\}$. All parts of this problem refer to these two sets.What is $|(A \cap B) \times (A \backslash B)|$?

(b) Define a function $f : B \to A$ by

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$$

for all $x \in B$. The symbol $\lfloor u \rfloor$ (the *floor* of u) means the largest integer less than or equal to the real number u, so for example $\lfloor 7.3 \rfloor = \lfloor 7 \rfloor = 7$. Is f one-to-one? Is f onto? Explain briefly.

(b) Let S denote the set of square positive integers, *i.e.,*

$$S = \{1, 4, 9, 16, \ldots, \}.$$

As usual, let $\mathbf{Q}$ denote the set of rational numbers. Is there an onto function $f : S \to \mathbf{Q}$? You may use any result presented in class or in the textbook to justify your answer.