A collection of problems from earlier midterm exams

CSCI2243-Logic and Computation

1.*Basic syntax and semantics of propositional formulas.* Construct the truth table for the formula displayed below. Then answer the three questions following it.

$$(p \to q) \oplus \neg(p \vee q).$$

(a) Is the formula a *tautology?* What is it about the truth table that tells you this?

(b) Is the formula *satisfiable?* What is it about the truth table that tells you this?

(c) Find an *equivalent* formula that is as simple as possible.

Solution. The truth table is depicted below.

p	q	$p \to q$	$p \vee q$	$\neg(p \vee q)$	$(p \to q) \oplus \neg(p \vee q)$
T	T	T	T	F	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	T	F

(a) The formula is **not a tautology**, because the value F appears in the right-hand column (that is, the formula is false for some assignments of truth values to the variables). *(b)* The formula **is satisfiable**, because the value T appears in the right-hand column (there are assignments to the variables that make the formula true). *(c)* The formula is true for exactly the assignments that make q true, and is thus equivalent to the formula q.

2. *Modeling with propositional formulas.* Twins Penelope and Quentin need to find seats at the dinner table, which has places for three people. Since Penelope is right-handed and Quentin is left-handed, **Quentin cannot sit immediately to the right of Penelope.** Since Penelope has very sensitive eyes, and seat 1 is opposite

a window, **Penelope cannot sit in seat 1.** Note that the diagram shows the table from above, so that seat 2 is to the **right** of seat 1.

Imagine giving the problem of determining an acceptable seating arrangement to a Satisfiability Solver. There are six boolean variables $p_1, p_2, p_3, q_1, q_2, q_3$ meaning 'Penelope sits in seat 1, *etc.*. The CNF specification of this simple problem has quite a few clauses.

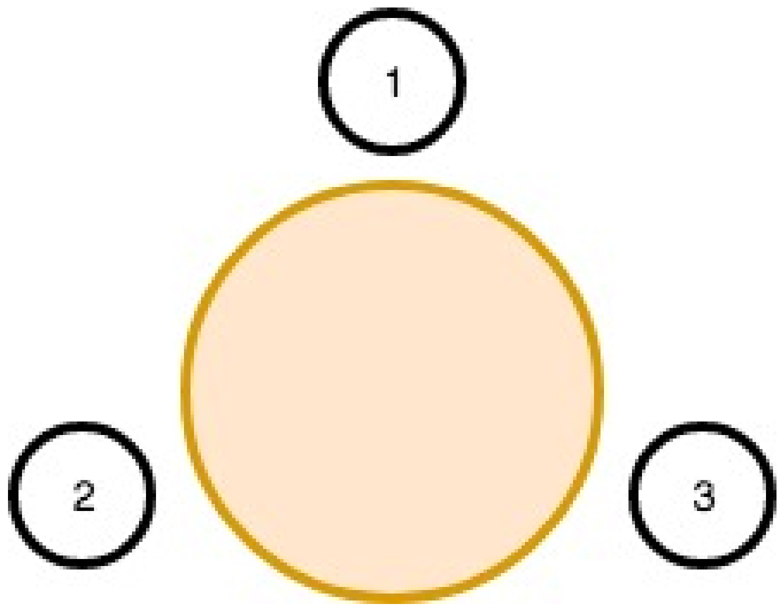

Figure 1: *Dinner table for Problem 2.*

(a) One of the clauses is

$$\neg p_1 \vee \neg q_1$$

and another is

$$\neg p_1 \vee \neg p_2.$$

Explain in English what these clauses say, and why they are part of the specification.

Solution. The first clause says that Penelope and Quentin are not both in seat 1. The second says that Penelope is not in both seat 1 and seat 2.

(b) Write a clause or clauses that express the first constraint; namely that Quentin cannot sit immediately to the right of Penelope.

Solution. This can be written as a conjunction of three clauses, ruling out the assignments $p_1 \wedge q_2$, $p_2 \wedge q_3$, $p_3 \wedge q_1$. The CNF encoding is

$$\neg p_1 \vee \neg q_2$$

$$\neg p_2 \vee \neg q_3$$

$$\neg p_3 \vee \neg q_1.$$

(c) Of course, you do not need a computer program to solve this problem. Write the truth values assigned to the variables $p_i, q_i,\ i = 1, 2, 3$, for *one* satisfying assignment. How many satisfying assignments are there altogether?

Solution. There are two satisfying assignments: p_2, q_1 true and the rest false: and p_3, q_2 true and the rest false.

4. *Sets and Functions* This problem refers to the sets and functions defined below. Each part of the question asks you either to identify a set, in which case you should list the elements, or a number.

$$A = \{0, 1, 2, 3\}$$
$$B = \{0, 2, 4, 6\}.$$
$$f : A \to B,$$

where $f(a) = 2|a - 2|$ for all $a \in A$.

$$g : B \to A$$

where $g(b) = b/2$ for all $b \in B$.
(a) $(B - A) \cup (A - B)$.
Solution. $\{1, 3, 4, 6\}$. (Union of $\{4, 6\}$ and $\{1, 3\}$.
(b) $|\mathcal{P}(A \cap B)|$.
Solution. 4. ($A \cap B = \{0, 2\}$, which has four subsets.)
(c) $|\mathrm{range}(g)|$. What does this tell you about whether g is one-to-one or onto?
Solution. 4. This tells you that g is both one-to-one and onto, since the range is the same size as the codomain (onto), and the same size as the domain (one-to-one). [Note that this sort of reasoning works only for finite sets—for infinite sets, the range can have the same cardinality as both the domain and codomain, while the function is neither one-to-one nor onto.]
(d) $g \circ f\,(3) + f \circ g\,(2)$.
Solution.

$$\begin{aligned} g(f(3)) + f(g(2)) &= g(2) + f(1) \\ &= 1 + 2 \\ &= 3. \end{aligned}$$

5. *Basic syntax and semantics of propositional formulas.* Consider the formula

$$(p \oplus q) \to (\neg p \wedge q).$$

(a) Write an equivalent formula in disjunctive normal form. You must justify your result: you may do this by means of a truth table, by algebraic manipulation of the formula, or by talking it through.

Solution. Truth tables are fine here. Just to mix things up a bit, let's see an algebraic argument, which is really just a translation of $\rightarrow$ and $\oplus$ into the familiar connectives.

$$\begin{aligned}(p \oplus q) \rightarrow (\neg p \wedge q) &\equiv \neg(p \oplus q) \vee (\neg p \wedge q) \\ &\equiv (p \wedge q) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q)\end{aligned}$$

(b) Find both a satisfying assignment and a non-satisfying assignment for this formula.

Solution. The only non-satisfying assignment is p true, q false; anything else is satisfying.

(c) Find an equivalent formula that uses only the connectives $\wedge, \vee, \neg$ and is as simple as possible (uses as few symbols as possible).

Solution. Our analysis above shows that $\neg(p \wedge \neg q)$ is equivalent to the formula. De Morgan's law gives an even simpler solution: $\neg p \vee q$. (In fact, this is equivalent to $p \rightarrow q$.)

(d) Let us interpret p as 'Emily orders soup' and q as 'Emily orders salad'. This cafeteria has no rules: You can order both soup and salad if you like; you can even have a sandwich. The following are two possible interpretations in English of our formula. Which of them are correct? (The answer could be 'both' or 'neither'.)

(i) Whenever Emily chooses between soup and salad (that is, whenever she has one but not both), she orders the salad and not the soup.

(ii) Emily never orders soup unless she gets salad, too.

Solution. They're both correct!

6. *Sets*

(a) Let $A = \{a, b, c\}$, $B = \{b, c, d\}$. List all the elements of $(A \cap B) \times (A \cup B)$.

Solution. $\{(b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d)\}$.

(b) Let A and B be arbitrary sets. Which of the following are true? Justify briefly.

$$A \cup B = (A - B) \cup B.$$

$$A \cup B = (A - B) \cup (B - A).$$

Solution. The first is true. I think a Venn diagram is simplest, but you can also note that it is equivalent to the propositional identity

$$p \vee q \equiv (p \wedge \neg q) \vee q.$$

To see that this really is an identity, use the distributive law to rewrite the right-hand side as

$$(p \vee q) \wedge (\neg q \vee q) \equiv (p \vee q) \wedge \mathbf{T} \equiv p \vee q.$$

(c) List all the elements of $\mathcal{P}(\mathcal{P}(\{1\}))$.

Solution. Since $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$, we get

$$\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}.$$

7. *Functions.*

(a) Figure 2 illustrates several assignments of elements of a set A to elements of A. Not all of these assignments define a function $f : A \to A$. Tell which of them are functions, and, in the cases where they are functions, whether these functions are one-to-one, onto, both, or neither.

Solution. The first and third are functions. The first is both one-to-one and onto, the second is neither.

(b) Let A and B be arbitrary sets. If $f : A \to B$ is one-to one and $g : B \to A$ is onto, if $g \circ f$ necessarily onto? The answer is no. Show this by giving a pictorial example, in the style of the diagrams in Figure 2.

Solution. (this counterexample is not in the form of a picture!) Let

$$A = \{1, 2, \}, B = \{1, 2, 3\}, f(1) = 1, f(2) = 2, g(1) = 1, g(2) = g(3) = 2.$$

You can verify (draw a picture!) that g is onto, but that $g \circ f$ is not .

(c) Let $A = \mathbf{Z}^+$, and let

$$B = \{x \in \mathbf{R} : 0 \leq x \leq 1\}.$$

Is there a one-to-one function $f : A \to B$? Is there an onto function $f : A \to B$? You may use any theorem you learned in the class to justify your answer.

Solution. There is a one-to-one function $A \to B$. This is just the same as saying that B has a countable subset, which is true of any infinite set. (If you want an explicit example, $f(j) = 1/j$ for all $j \in \mathbf{Z}^+$ does it, but this is not really necessary.)

There is no onto function $A \to B$. The short reason is that B is uncountable and A is countable. A one-to-one function would give us a list $r_1, r_2, \ldots$ that contains every real number between 0 and 1.

8. *Propositional Logic.*

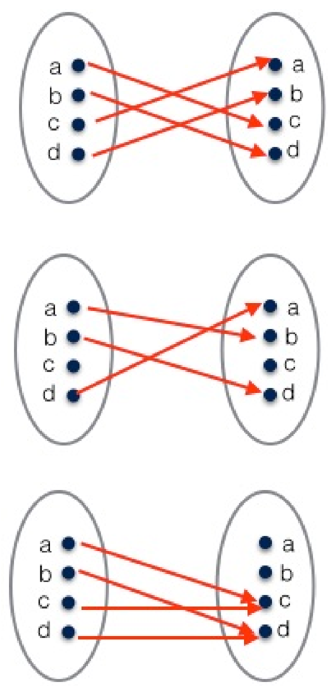

Figure 2: The three relations for Problem 7.

(a) Write a sentence of propositional logic in *conjunctive normal form* that is equivalent to

$$p \rightarrow (q \wedge r).$$

Solution. We'll use algebra here.

$$\begin{aligned} p \rightarrow (q \wedge r) &\equiv \neg p \vee (q \wedge r) \\ &\equiv (\neg p \vee q) \wedge (\neg p \vee r) \end{aligned}$$

(b) How many satisfying assignments does the formula in (a) have? Explain briefly.

Solution. If p is false, and then any of the four assignments of values to q and r will work. If p is true then both q and r must be true. So there are five satisfying assignments in all.

(c) Find a formula that is as simple as possible (uses as few symbols as possible) that is equivalent to

$$(\neg p \vee q) \leftrightarrow p.$$

You may use any method you like—algebra, truth tables, or just talking it through, but you must have some explanation of your reasoning.

Solution. Let's talk it out. What does it take to make the formula true? If p is true, then the left-hand side must be true, so q has to be true. If p is false, then the left-hand side must be false to make the biconditional true, however if p is false then $\neg p \vee q$ is true. So the only solution is to have p and q true, thus this is equivalent to $p \wedge q$.

9. *Sets*

(a) Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{4, 5, 6, 7, 8, 9, 10\}$. All parts of this problem refer to these two sets.What is $|(A \cap B) \times (A \backslash B)|$?

(b) Define a function $f : B \rightarrow A$ by

$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$$

for all $x \in B$. The symbol $\lfloor u \rfloor$ (the *floor* of u) means the largest integer less than or equal to the real number u, so for example $\lfloor 7.3 \rfloor = \lfloor 7 \rfloor = 7$. Is f one-to-one? Is f onto? Explain briefly.

Solution. f is not one-to-one because $f(4) = f(5) = 2$. f is not onto because neither 1 nor 6 is in the range of f.

(b) Let S denote the set of square positive integers, *i.e.,*

$$S = \{1, 4, 9, 16, \ldots, \}.$$

As usual, let **Q** denote the set of rational numbers. Is there an onto function $f : S \to \mathbf{Q}$? You may use any result presented in class or in the textbook to justify your answer.

Solution. Yes such a function exists. Both sets are countable, as we learned in class, so there is a bijection between them, and in particular an onto function from S to **Q**. You do not have to exhibit such a function, but if you want to be more explicit, recall that we can create a list that contains all the rational numbers:

$$1/1, -1/1, 1/2, -1/2, 2/1, -2/1, 1/3, -1/3, 2/2, / - 2/2, 3/1, \ldots$$

We can then map n^2 to the n^{th} element on the list.