# Lecture 12: More on the Exponential Distribution

CSCI2244-Randomness and Computation

April 1, 2019

## 1 Calculation of the expected value.

Well, I figured I'd do this right, since it wasn't super-clear in class!

Recall that the expected value of a continuous random variable $X$ is

$$\int_{-\infty}^{\infty} t f_X(t)dt,$$

provided this integral is well defined (*i.e.,* this improper integral has to converge to a finite value). In the case of the exponential distribution, the density $f_X(t)$ is 0 for negative $t$, so the expected value is given by

$$\int_0^{\infty} t \cdot \lambda e^{-\lambda t} dt.$$

To evaluate the indefinite integral, we use integration by parts: Set

$$u = t, dv = \lambda e^{-\lambda t} dt.$$

This gives

$$du = dt, v = -e^{-\lambda t}.$$

Thus an antiderivative is given by

$$\begin{aligned} uv - \int v du &= -te^{-\lambda t} + \int e^{-\lambda t} dt \\ &= -te^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t} \end{aligned}$$

We now have to evaluate these between 0 and $\infty$. The first term $-te^{-\lambda t}$ is of course 0 at $t = 0$. What about its value as $t$ approaches $\infty$? The factor $t$ increases without bound, while $e^{-\lambda t}$ approaches 0. What about their product? The rule here is that the exponential function $e^{\lambda t}$ grows faster than any polynomial function, like $t$, so $t/e^{\lambda t}$ approaches 0. So we can throw out that first term.

When we evaluate the second term $-\frac{1}{\lambda}e^{-\lambda t}$ at $\infty$, we get 0, at at 0 we get $-\frac{1}{\lambda}$, so we conclude that the integral is $\frac{1}{\lambda}$, as we stated earlier.

# 2 Worked example

For example, suppose the arrival of the next customer in a store is given by a random variable $X$ with an exponential distribution, and that the average time between arrivals is 5 minutes. Then $\frac{1}{\lambda} = 5$, so $\lambda = 0.2$. We can then answer the following kind of question: What is the probability that no customer will enter the store in then next 7 minutes?

We are asking for $P(X > 7)$. This is

$$1 - P(X \leq 7) = 1 - (1 - e^{-\lambda \cdot 7}) = e^{-0.2 \times 7} \approx 0.25.$$

In general, the probability that no one arrives in the next $t$ minutes is $e^{-\lambda t}$.

# 3 Connection with the Poisson Distribution

What if we asked for the probability that *exactly* one person arrives in the next $t$ units of time? Or exactly 2? or 4? This is actually a tricky problem, and solving it rigorously requires some multivariable calculus. But here is the answer: The probability of exactly $k$ arrivals in $t$ units of time is

$$\frac{(\lambda t)^k}{k!}e^{-\lambda t}.$$

With $t = 1$ (number of arrivals in 1 unit of time), this is the Poisson distribution.

Observe that earlier we saw the Poisson distribution as an approximation to the binomial distribution when the number of trials was large and the probability of success small. Here we obtain it as the exact answer to a problem about processes with the exponential distribution.