Lecture 4: Binomial Coefficients

January 31, 2019

1 Definition

The number of k-element subsets of an n-element set (when $0 \leq k \leq n$);

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

pronounced 'n choose k'. Observe that this formula works even for the cases where $n = 0$, or where $k = 0$ or $k = n$, because $0! = 1$.

If you have to compute it, *don't* compute large factorials. For example, the number of 5-card selections from a standard deck is

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1},$$

and the right-hand side is more reasonable to compute. (It's 2,598,960.)

2 Basic properties.

We'll skip the simple proofs. There are at least two ways to prove every one of these: Using the formula and some algebra, or using a set counting argument.

$\binom{n}{0} = \binom{n}{n}$ for all $n \geq 0$.

$\binom{n}{k} = \binom{n}{n-k}$ for all $0 \leq k \leq n$.

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for all $1 \leq k \leq n$.

The first equation is just a special case of the second. The first and third formulas allow you to construct, row by row, the table of binomial coefficients called *Pascal's triangle*. In the table below, the rows are indexed by n, the columns by k, and both indices begin at 0.

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

3 Poker odds

Select five cards from a 52-card deck. What is the probability of getting a flush (all cards of the same suit)? What is the probability of getting a full house (three cards of one rank, two of a different rank)? of a pair (two cards of the same rank and the remaining cards of three different ranks)?

We represent each outcome as a set of 5 cards from a standard deck. We make the assumption (justified by a common-sense argument, but also borne out by experiment) that no subset is more likely than any other, so that the probability distribution is uniform, and thus the probability of any event E is given by

$$\frac{|E|}{\binom{52}{5}}.$$

So the problem reduces to counting the number of elements in E.

For the flush, each suit has 13 cards, so that there are $\binom{13}{5}$ flushes in that suit, and thus

$$|E| = 4 \cdot \binom{13}{5}.$$

For the full house, we specify the full house by choosing (a) the rank in which the triple will appear; (b) the three cards in this rank that make up the triple; (c) the rank in which the pair will appear; (d) the two cards in this rank that make up the pair. We get, by the multiplication principle,

$$|E| = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2}.$$

For the pair, we choose (a) the rank in which the pair appears; (b) which two cards in this rank make up the pair; (c) a set of three different ranks from the 12 remaining; (d) for each of these three ranks, one of the four cards in the rank. This gives

$$|E| = 13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3.$$

These can get quite tricky, and it is easy to make the error of overcounting.

4 Counting sequences

How many sequences of 8 H's and T's are there in which there are exactly 3 H's? There is a simple one-to-one correspondence between the set of such sequences and the set of three-element subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$:

$$\begin{array}{rcl} \{2,4,7\} & \leftrightarrow & THTHTTHT \\ \{1,6,8\} & \leftrightarrow & HTTTTHTH. \end{array}$$

Of course this idea works for any length sequence and any number of heads. So the number of sequences of n H's and T's (or x's and y's, or 0's and 1's) with exactly k H's is $\binom{n}{k}$.

4.1 Binomial theorem.

$$(x+y)^4 = xxxx + xxxy + xxyx + \cdots + yyyx + yyyy.$$

The right-hand side is the sum of all sequences of 4 x's and y's. The coefficient of $x^{4-k}y^k$ is the number of such sequences containing exactly k y's. Thus

$$\begin{aligned} (x+y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4. \end{aligned}$$

By the same argument, for any $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

4.2 MISSISSIPPI

How many ways are there to arrange the letters of the word MISSISSIPPI? There are 11 letters, and if they were all different, the answer would be $11!$. But here, many of these $11!$ arrangements represent the same string. Here is a solution, where we treat the problem of specifying an arrangement as a sequence of choices: We need to choose 4 out of the 11 positions to hold the I's, then for each such choice, 4 out of the remaining 7 positions to hold the S's, then 2 out of the remaining 3 positions for the P's. After this, only one position is left open, for the M. So the number is

$$\binom{11}{4}\binom{7}{4}\binom{3}{2} = \frac{11!}{4!7!} \cdot \frac{7!}{4!3!} \cdot \frac{3!}{2!1!} = \frac{11!}{4!4!2!}.$$

By the way, this last value is sometimes written as

$$\binom{11}{4, 4, 2, 1}$$

and is called a *multinomial coefficient*. It is the coefficient of $w^4x^4y^2z$ in the expansion of $(w + x + y + z)^{11}$.

4.3 Coin tosses

When we toss a fair coin n times, the 2^n sequences of n H's and T's are equally likely. So the probability of getting exactly k heads on n tosses is

$$\binom{n}{k}2^{-n}.$$

For an unfair coin where the heads probability is p, the probability of any fixed sequence of k H's and $n - k$ T's

$$p^k(1 - p)^{n-k},$$

so the probability of getting exactly k heads on n tosses is

$$\binom{n}{k} \cdot p^k(1 - p)^{n-k}.$$

Observe that when $p = \frac{1}{2}$, this formula gives the same result as for the fair coin.

For example, suppose we toss a fair coin 10 times. What is the probability of getting at least 7 heads? By the above this is

$$2^{-10} \cdot \left(\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} \right) \approx 0.172.$$

For a fair coin, the probability of getting at least 7 tails is the same, so this means that $1 - 2 \times 0.172 = 0.656$ of the time, the number of heads will be 4,5,or 6.

Suppose instead that the coin is unfair, and the heads probability is 0.6. Then the probability of at least 7 heads is

$$\sum_{k=7}^{10} \binom{10}{k} 0.6^k 0.4^{10-k} \approx 0.382.$$

The probability of getting at least 7 tails is given by the same formula with 0.4 and 0.6 interchanged. This is approximately 0.055.

5 An application of Stirling's formula

You may remember from our simulation results that when we tossed a fair coin an even number $2n$ of times, getting n heads is the most likely result, but the probability of getting *exactly* n heads gets smaller and smaller as n increases. What exactly is the relation between this probability and n?

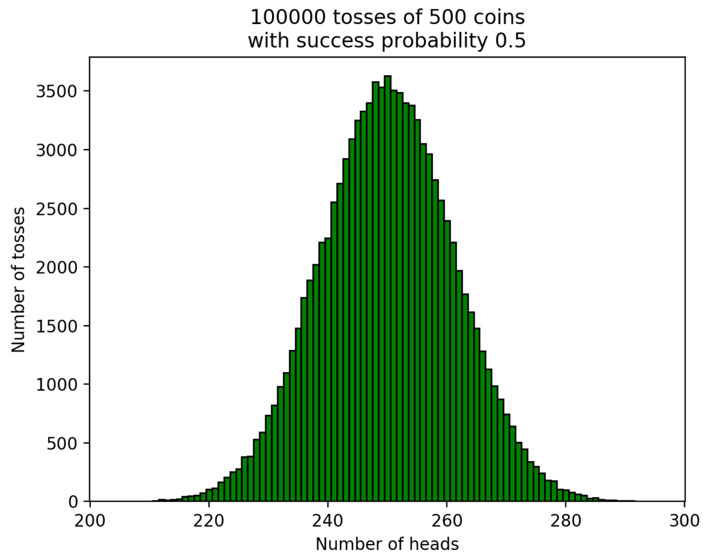
We can use Striling's formula fo $n!$ to estimate this. The probability is

$$\begin{aligned} \binom{2n}{n} \cdot 2^{-2n} &\approx \frac{(2n)!}{n! \cdot n!} \cdot 2^{-2n} \\ &= \frac{(2n)^{2n}}{e^{2n}} \cdot \left(\frac{e^n}{n^n} \right)^2 \cdot \sqrt{2\pi \cdot 2n} \cdot \frac{1}{2\pi n} \cdot 2^{-2n}, \end{aligned}$$

which is quite awful-looking, but as it turns out practically everything cancels, and at the end you're just left with

$$\frac{1}{\sqrt{\pi n}}.$$

Let's compare this to our simulation results. The plot below shows the result of simulating 500 tosses of a fair coin 100,000 times. The height of the plot is



3633, corresponding to a probability of 0.0363. We have $n = 250$, so

$$\frac{1}{\sqrt{\pi n}} = 0.035682,$$

which is consistent with the theoretical result.