CSCI3390-Second Test

April 26, 2016

Each of the 15 parts of the problems below is worth 10 points, except for the more involved 4(d), which is worth 20. A perfect score is 100: if your score is in excess of 100, I will treat the additional points as extra credit to be added to either homework or test scores. These are independent of one another, so you can work them in any order, but make sure you label clearly which question you are answering!

1. Let Σ be a finite alphabet. Let $L \subseteq \Sigma^*$, where $L \in \mathbf{NP}$. Answer the following questions 'True', 'False', 'Believed True', or 'Believed False'. The last two categories are to be used if the answer depends on the truth of a widely-believed, but not-yet-proved, conjecture.

   (a) Every such L is Turing-recognizable.

   (b) Every such L is decided by a deterministic TM that takes $O(n^k)$ steps on inputs of length n. (Here and in the subsequent parts $k > 0$ is a constant that depends on the TM but not on n.)

   (c) **Some** such L is decided by a deterministic TM that takes $O(n^k)$ steps on all inputs of length n.

   (d) Every such L is decided by a deterministic TM that takes $O(2^{n^k})$ steps on all inputs of length n.

2. Let $COMPOSITE \subseteq \{0, 1, \ldots, 8, 9\}^*$ be the set of decimal representations of composite positive integers. That is, L is the set of strings

$$\{4, 6, 8, 9, 10, 12, 14, 15, \ldots\}.$$

   (a) Give a simple proof, involving no special facts from number theory, that $COMPOSITE \in \mathbf{NP}$.

(b) The following is a 'proof' that $COMPOSITE \in \mathbf{P}$: Given input N, divide N in turn by $2, 3, \ldots, N-1$. Answer 'Yes' and halt if the remainder is zero, and answer 'No' otherwise. Since division of two integers takes quadratic time, and we perform at most $N-2$ divisions, this algorithm takes time polynomial in the size of the input. This argument is incorrect—what is the error?

(c) So, is $COMPOSITE$ in **P**? You don't have to prove your answer, you can simply cite theorems that were discussed in class.

3. These problems concern boolean satisfiability.

(a) Consider the following propositional formula:

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r) \wedge (\neg q).$$

Is this formula satisfiable? If so, give *all* satisfying assignments. (It may helpful to note that the first four clauses exhibit a particular pattern that we've seen before.)

(b) In class we showed that 2-SAT is in **P**. The formula above is not a legal input to 2-SAT, since the first clause contains more than two literals. Show nonetheless, satisfiability for propositional formulas that contain *at most one* clause with more than 2 literals is in **P**. (HINT: You don't have to tell me the algorithm for 2-SAT, but you need to call this algorithm.)

(c) A boolean formula is a *tautology* if **every** assignment of truth values to variables is a satisfying assignment. Is the problem of determining whether a boolean formula in CNF is a tautology in **P**? Does the answer depend on the status of the $\mathbf{P} = \mathbf{NP}$ conjecture?

(d) A boolean formula is a *contradiction* if **no** assignment of truth values to variables is a satisfying assignment. Is the problem of determining whether a boolean formula is a contradiction in **P**? Does the answer depend on the status of the $\mathbf{P} = \mathbf{NP}$ conjecture?

4. These problems concern the Hamiltonian Path problems for both directed and undirected graphs, and require you to correctly identify polynomial-time reductions. Here, to remind you, are the problems:

   DIRECTED HAMILTONIAN PATH

   Input: A directed graph G and two vertices s and t.

   Output: Yes if and only if there is a directed path from s to t that visits each vertex exactly once.

   UNDIRECTED HAMILTONIAN PATH

   The description is the same, except both the graph G and the required path are undirected.

   The first two parts of the problem involve the following two constructions on graphs:

   Construction 1:

   Starting from an undirected graph G, produce a directed graph G' with the same set of vertices by replacing each undirected edge $\{i, j\}$ by a pair of directed edges (i, j) and (j, i).

   Construction 2:

   Starting from a directed graph H produce an undirected graph H' with the same set of vertices by replacing each directed edge (i, j) where $i \neq j$ by the undirected edge $\{i, j\}$.

   The constructions are illustrated in the accompanying figure. Note that an undirected graph cannot have multiple edges between the same two vertices, nor loops at a single vertex, so if H contains a pair of edges (i, j) and (j, i), H' will only have $\{i, j\}$, and if H contains loops (i, i), these will not appear in H' at all.

   (a) Choose which one of the following answers is appropriate, and fill in the blanks:

      Construction 1 is a polynomial-time reduction from [PROBLEM] to [PROBLEM]. If we already know [PROBLEM] is **NP**-hard, this reduction proves [PROBLEM] is $\mathbf{NP}$-hard.

      OR

Construction 1 is not a polynomial-time reduction from either of these problems to the other because.....

(b) Answer the above question for Construction 2.

(c) The *degree* of a vertex in an undirected graph is the number of neighbors it has. (See the caption of the accompanying figure.) Prove that if we restrict to graphs in which every vertex has degree at most 2, then **UNDIRECTED HAMILTONIAN PATH** is in **P**.

(d) (Harder.) What if we restrict to graphs in which every vertex has degree at most 3? Show that this problem is **NP**-complete. (You may have to remember some proofs we did in class.)

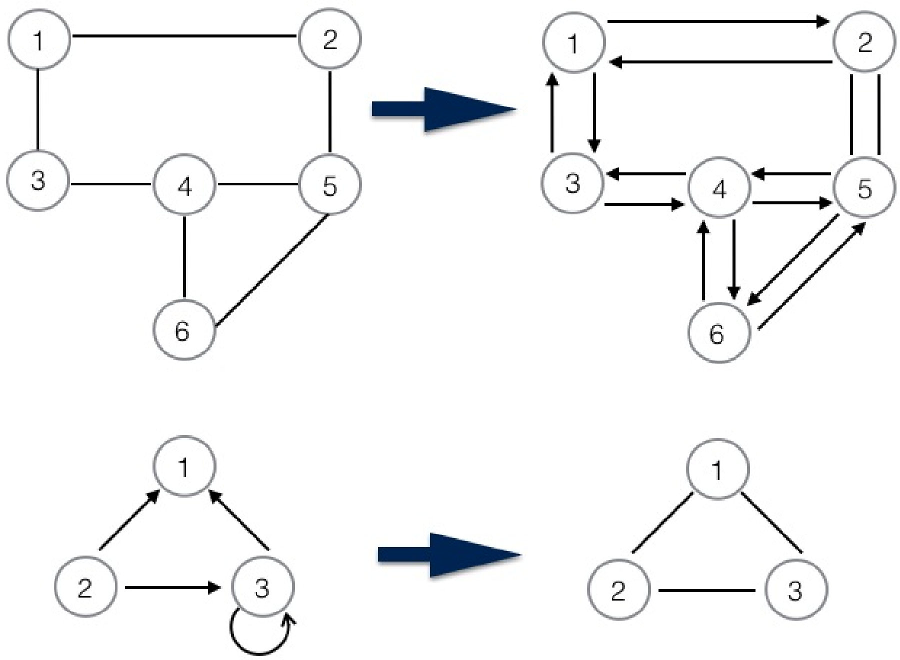

Figure 1: Construction 1 (top) and construction 2(bottom). In the top left diagram, vertices 1,2,3 and 6 all have degree 2, vertices 4 and 5 have degree 3. All vertices in the lower right have degree 2.